Solutions to Nonsimilar Energy and Species Equations in Falkner-Skan Flows

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Solutions involving the energy and species equations for flows with velocities described by the Falkner-Skan equation are presented. The sets of eigenvalues and eigenfunctions for various β arising from the problem of arbitrary initial distributions of stagnation enthalpy are calculated. Exact solutions in the cases of step function, as well as arbitrary distributions of wall enthalpy are obtained for constant $\rho\mu$ and Prandtl number of unity. Application is also made to the problem of reactant concentrations in the diffusion-controlled flow over a catalytic surface by constructing an integral equation that can be solved numerically.

Introduction

CONSIDERATION of the distribution of either the energy or species conservation in laminar boundary layers the velocity distributions of which are simply described has led to interesting calculations. Thus Fox and Libby (Ref. 1) provided solutions to the energy equation for boundary layers with velocities given by the Blasius solution without mass transfer; these solutions have been used in Refs. 2 and 3 to solve problems of surface-catalyzed reactions and in Refs. 4 and 5 to solve problems involving the energy distribution with arbitrarily prescribed surface temperature and heat transfer, respectively. Finally, Wallace and Kemp (Ref. 6) have extended these considerations to the case of mass transfer by treating the species equation for cases in which the velocity field is described by the Blasius equation with mass transfer.

Presented here is an extension of these ideas to boundary layers with velocity distributions described by the Falkner-Skan equation and thus with streamwise pressure gradient of the similarity type. In problems wherein the pressure gradient cannot be neglected, the velocity field is coupled to the energy and species fields through the variation of mass density. However, if the flow is almost incompressible, i.e., if the stagnation enthalpy deviates only slightly from that of external stream, and if the composition is nearly uniform, the first-order momentum equation, with the product of mass density and viscosity coefficient ($\rho\mu$) being constant, can be uncoupled from the energy equation; thus, the problem, which is the one considered here, becomes much easier to solve.

Two problems are treated; there is first presented the description of a flow, which has a velocity field described by the Falkner-Skan equation, has a distribution of stagnation enthalpy arbitrarily specified at a given streamwise station, and which has a constant wall enthalpy downstream of that initial station. Assuming $C \equiv (\rho\mu/\rho_e\mu_e) = 1$ and the Prandtl number $\sigma = 1$, we show that the solution to this problem can be given in terms of orthogonal sets of eigenfunctions which can be applied to a variety of problems. For example, the perturbation due to deviations of C, σ, and f can be determined in terms of a Green's function constructed with those eigenfunctions and the problem with the arbitrarily specified wall enthalpy can be solved by a Duhamel integral involving these same eigenfunctions. These applications have been described clearly in Refs. 1 and 2 in connection with flows without pressure gradients.

Because of the relation among species conservation with no gas-phase chemical reaction, element conservation, and energy conservation expressed in terms of the stagnation enthalpy, the eigenvalues and eigenfunctions of the energy equation can also be applied directly to problems related to flows with chemical reactions. Thus the second problem we consider is an application of these eigenfunctions to the species equation, in particular to the problem that has been treated by Rae[7] and which involves the reactant concentrations in the diffusion-controlled flow over a catalytic surface with the velocities described by the Falkner-Skan equation. The solution for the concentration field is given in terms of an integral equation that can be calculated numerically. The results are compared and show good agreement with those of Ref. 7.

This second application to the boundary layer with surface reaction and streamwise pressure gradient follows previous work by Chung, Liu, and Mirels,[8] Inger,[9] Rosner,[10] and Freeman and Simpkins.[11] In most of these references techniques involving power series expansions have been used. Our eigenfunction approach has the advantage of, on the one hand, being able to handle cases to which power series are applicable and, on the other, to handle more general cases.

We remark here as to the relation of the present work to other methods of treating the same problems. Since we are dealing with nonsimilar conservation equations for energy or species with a simply described velocity field, direct solution by finite-difference or difference-differential method can be achieved. Such solutions are in principal exact. It is also possible to obtain approximate solutions by integral methods, either of the simple, zero moment or of the multiple moment type. The present eigenvalue approach should be considered intermediate in complexity and accuracy to the aforementioned methods. In principal, it is exact within certain approximations relative to thermodynamic and transport properties, but because the eigenfunction representation is truncated after a number of terms, the results are in fact approximate. Previous experience with boundary-layer flows with velocities described by the Blasius solution suggests that the present approach has wide applicability.

Received November 27, 1968; revision received April 21, 1969. This paper was taken from a part of the author's dissertation submitted to the Faculty of the University of California, San Diego, in partial fulfillment of the requirements for the Ph. D. degree (1968). This research was supported by the Advanced Research Projects Agency (Project DEFENDER) under Contract DA-31-124-ARO-D-257, monitored by the U. S. Army Research Office—Durham. The author is pleased to acknowledge P. A. Libby for suggesting this problem and to thank him for his continuous guidance and correction throughout this research.

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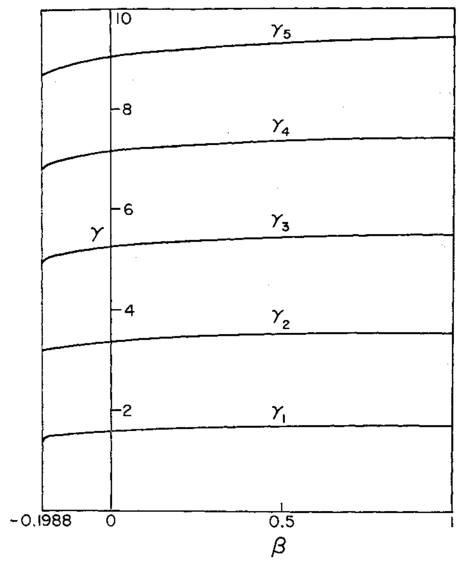

Fig. 1 The variation of the first five eigenvalues with β.

Initial Profile Problem

The energy equation for a laminar boundary layer with the ratio of density viscosity product $C \equiv (\rho\mu/\rho_e\mu_e) = 1$ and the Prandtl number $\sigma = 1$ can be written in terms of the Levy-Lees variable η and s, and of the stagnation enthalpy ratio $g = h_s/h_{s,e}$ as

$$g_{\eta\eta} + fg_\eta = 2s(f_\eta g_s - f_s g_\eta) \quad (1)$$

where

$$\eta = \frac{\rho_e u_e r^j}{2s} \int_0^y \frac{\rho}{\rho_e}\, dy$$

$$s = \int_0^x \rho_e\mu_e u_e r^{2j} \cdot dx \quad (2)$$

where $j = 0, 1$ for two-dimensional and axisymmetric flows, respectively.

We first consider a flow with velocities described by the Falkner-Skan equation, i.e.,

$$f'''_0 + f_0 f''_0 + \beta(1 - f'^2_0) = 0 \quad (3)$$

subject to the boundary conditions

$$f_0(0) = f'_0(0) = 0, f'_0(\infty) = 1$$

where $\beta = (2s/u_e)(du_e/ds)$, for an incompressible, laminar boundary layer and $\beta = (2s/u_e)(du_e/ds)(h_{s,e}/h_e)$, for a compressible, laminar boundary layer under similarity conditions if the stagnation enthalpy across the layer does not deviate significantly from its external value. Since $f_0(\eta)$ is independent of s, Eq. (1) can be rewritten as

$$g_{\eta\eta} + f_0 g_\eta - 2sf_{0\eta} g_s = 0 \quad (4)$$

If the distribution of stagnation enthalpy at a given streamwise station, say $s = s_i \neq 0$, is specified and if the enthalpy distribution on the wall downstream of the initial station is constant, the initial and boundary conditions of the energy equation (4) can be written as

$$g(s_i, \eta) = G(\eta),\ g(s,0) = g_w = \text{const} \quad (5)$$

and $g(s, \infty) = 1$. To find a solution let

$$g(s,\eta) = g_0(\eta) + g_1(s,\eta) \quad (6)$$

where g_0 satisfies

$$g''_0 + f_0 g'_0 = 0 \quad (7)$$

subject to the boundary conditions $g_0(0) = g_w$, $g_0(\infty) = 1$. The solution of g_0 can be written in terms of a more general function

$$\bar{F} = \int_0^\eta \exp\left(-\int_0^\eta f_0(\bar{\eta})d\bar{\eta}\right) d\eta \Big/ \left[\int_0^\infty \exp\left(-\int_0 f_0(\bar{\eta})d\bar{\eta}\right) d\eta\right] \quad (8)\dagger$$

such that

$$g_0(\eta) = (1 - g_w)\bar{F}(\eta) + g_w$$

Substitution of Eq. (6) into Eq. (4) yields

$$g_{1\eta\eta} + f_0 g_{1\eta} - 2sf'_0 g_{1s} = 0 \quad (9)$$

subject to the boundary conditions

$$g_1(s_i,\eta) = \tilde{G}(\eta) = G(\eta) - g_0(\eta),\ g_1(s,0) = 0,\ g_1(s,\infty) = 0$$

By the method of separation of variables, i.e., $g_1(s,\eta) = S(s)M(\eta)$, we find $S(s) \sim s^{-\gamma/2}$ and

$$M'' + f_0 M' + \gamma f'_0 M = 0 \quad (10)$$

subject to the boundary conditions $M(0) = M(\infty) = 0$, where γ is the separation parameter. Clearly, Eq. (10) defines eigenvalues γ and eigenfunctions M. If the eigenfunctions are required to decay exponentially, the eigenvalues γ are found to be discrete. Moreover, Eq. (10) can be written in a typical Sturm-Lionville form,

$$\left[\exp\left(\int_0^\eta f_0 d\bar{\eta}\right) M'\right]' + \gamma f'_0 \exp\left(\int_0^\eta f_0 d\eta\right) M = 0 \quad (11)$$

By the usual means, the eigenfunctions can be shown to be real and to be orthogonal according to

$$\int_0^\infty f'_0 \exp\left(\int_0^\eta f_0 d\bar{\eta}\right) M_m M_n d\eta = C_n \delta_{m,n} \quad (12)$$

provided that M_m decays exponentially for large η. The solution for g_1 is thus given by

$$g_1 = \sum_{n=1}^{\infty} A_n \left(\frac{s}{s_i}\right)^{-\gamma n/2} M_n(\eta) \quad (13)$$

and

$$A_n = \frac{1}{C_n} \int_0^\infty f'_0 \exp\left(\int_0^\eta f_0 d\eta\right) \tilde{G}(\eta) M_n(\eta) d\eta \quad (14)$$

Numerical Results of the Eigenvalue Problem

As long as $f'_0 > 0$ for $\eta > 0$, it can be shown that the eigenvalues are positive. If f'_0 changes sign in some range of η, as for example in the lower branch solutions of the Falkner-Skan equation, there exists an infinite set of real eigenvalues that have limit points $+\infty$ and $-\infty$.‡[12] For $\beta = \beta_0 = -0.1988377$, the eigenfunction corresponding to the lowest eigenvalue can be obtained in closed form such that

$$M_1(\eta) = -f''_0(\eta)/\beta_0 = 5.029227 f''_0(\eta)$$

subject to the scaling condition $M'_1(0) = 1$, with the corresponding eigenvalue

$$\lambda_1 = 1 - 2\beta_0 = 1.397654$$

The other eigenvalues and eigenfunctions have to be calculated by a numerical procedure, e.g., by the quasilinear scheme discussed in Ref. 14. The numerical results of the first ten eigenvalues and their normalization constant are listed in Table 1 for $\beta = -0.19$, 0, 0.4, and 0.5. The first

† The numerical values of $\bar{F}'(0)$ for various β are as follows: for $\beta = -0.19$; 0; 0.4; 0.5; and 1, $\bar{F}'(0) = 0.3650$; 0.4696; 0.5300; 0.5390; and 0.5705, respectively.

‡ In Ref. 13 there is a detailed discussion of the eigenvalues for Falkner-Skan flows.

five eigenvalues vs β corresponding to the upper solution branch of the Falkner-Skan equation are plotted in Fig. 1. Since all the eigenvalues are positive, the solutions of the stagnation enthalpy in Eq. (4) are spatially stable if the velocities are described by the upper branch solutions of the Falkner-Skan equation. As to those corresponding to the lower branch solutions, there are infinite numbers of negative eigenvalues and the solutions to Eq. (4) are spatially unstable.

The terms spatially stable and spatially unstable, as used here, refer to the behavior far downstream of the station at which the initial data are specified. Consider Eqs. (13) and (14); we see that if the initial data result in the presence of a particular eigenfunction $M_n(\eta)$ with $\gamma_n > 0$ then, with increasing (s/s_i), the effect of that contribution to the downstream energy field will decay. In this case we consider the flow to be spatially stable to such a perturbation; if all eigenvalues are positive, then the flow is stable to all perturbations in initial data. The converse prevails if $\gamma_n < 0$ for some n. Then ideas follow the considerations of Chen and Libby (Ref. 13) for velocity fields in Falkner-Skan flows.

Application to Some Heat-Transfer Problems

The sets of eigenfunctions developed above can be used directly to solve the problem in which a surface exposed to a steady flow with external velocity $u_e \sim \chi^m$ has a wall enthalpy $g_{w,1}$ upstream of $s = s_i$ and a wall enthalpy $g_{w,0}$ downstream of s_i, where $g_{w,1}$, $g_{w,0}$ are constant. This problem has been treated by Eckert[15] and by Fox and Libby[1] in the case of a flat plate, i.e., $\beta = 0$. We consider the extension to the case of constant $\beta \neq 0$.

There is a similar solution for $s < s_i$ so that the initial profile $G(\eta)$ can be written as

$$G(\eta) = g_{w,1} + (1 - g_{w,1})\tilde{F}(\eta)$$

whereas the stagnation enthalpy far downstream is expected to be another similar solution,

$$g(\infty,\eta) = g_\infty(\eta) = g_{w,0} + (1 - g_{w,0})\tilde{F}(\eta)$$

where $\tilde{F}(\eta)$ is given by Eq. (8).

The coefficient A_n can be evaluated from Eq. (14) once and for all in the form

$$\tilde{A}_n \equiv \frac{A_n}{g_{w,1} - g_{w,0}} = \int_0^\infty f'_0 \exp\left(\int_0^\eta f_0 d\eta\right) \frac{(1 - \tilde{F})M_n d\eta}{C_n}$$

which can be easily shown to be

$$\tilde{A}_n = (\gamma_n C_n)^{-1} \tag{15}$$

The solution for g for $s > s_i$ is readily obtained from Eqs. (6) and (13).

The numerical values of the eigenvalues and their normalization constants in Table 1 can be used directly to evaluate the heat transfer to the wall downstream of the discontinuity, i.e.,

$$g'(s,0) = (g_{w,0} - 1)\tilde{F}'_w + (g_{w,1} - g_{w,0}) \sum_{n=1}^{\infty} \frac{1}{C_n\gamma_n}\left(\frac{s}{s_i}\right)^{-\gamma n/2}$$

Table 1 The first ten eigenvalues and squares of norms for various values of β

	β							
	−0.19		0		0.4		0.5	
n	γn	Cn	γn	Cn	γn	Cn	γn	Cn
1	1.448	5.726	1.573	3.437	1.642	2.665	1.652	2.571
2	3.187	4.013	3.387	2.408	3.504	1.904	3.521	1.842
3	5.001	3.383	5.255	2.014	5.406	1.603	5.428	1.553
4	6.848	3.007	7.145	1.791	7.326	1.431	7.352	1.387
5	8.714	2.798	9.051	1.635	9.286	1.334	9.286	1.291
6	10.59	2.583	10.96	1.531	11.19	1.229	11.23	1.191
7	12.49	2.430	12.88	1.436	13.14	1.191	13.17	1.152
8	14.38	2.327	14.81	1.375	15.08	1.106	15.12	1.073
9	16.29	2.218	16.74	1.306	17.03	1.090	17.08	1.054
10	18.20	2.145	18.68	1.264	18.99	1.019	19.03	0.988

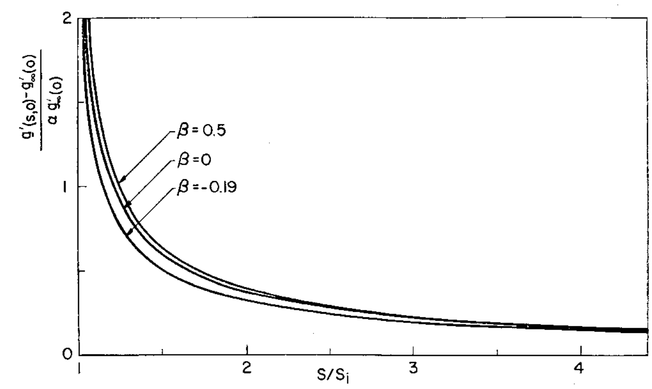

Fig. 2 Distribution of heat transfer for step-function wall enthalpy.

The quantities

$$\frac{g'(s,0) - g'_\infty(0)}{\alpha g'_\infty(0)} = \frac{1}{\tilde{F}'_w} \sum_{n=1}^{\infty} \frac{1}{C_n\gamma_n}\left(\frac{s}{s_i}\right)^{-\gamma n/2} \tag{16}$$

are plotted in Fig. 2 as functions of s for various β, where $\alpha = (g_{w,1} - g_{w,0})/(1 - g_{w,0})$. We have employed the first ten terms of the infinite series on the right-hand side of Eq. (16) and further applied the nonlinear transformation of Shanks[16] to improve convergence. The data of the curve for $\beta = 0$ are taken from those of Ref. 4. In terms of the representation of Fig. 2, we note surprisingly little effect of pressure gradient.

Our eigenfunctions can also be applied to the problem of heat transfer to a surface with an arbitrarily specified wall enthalpy. Consider $f \equiv f_0$ and let

$$g(s,0) = \begin{cases} g_w(s_i) = \text{const}, & (0 \leq s \leq s_i) \\ g_w(s), & (s_i < s) \end{cases}$$

To get the solution we construct first a step-function solution $\bar{g}(s,\eta;\xi)$ such that

$$(\partial^2\bar{g}/\partial\eta^2) + f_0(\partial\bar{g}/\partial\eta) - 2sf'_0(\partial\bar{g}/\partial s) = 0 \tag{17}$$

$$\bar{g}(0,\eta;\xi) = \bar{g}(s,\infty;\xi) = 0$$

$$\bar{g}(s,0;\xi) = 0 \qquad (0 \leq s < \xi)$$

$$= 1 \qquad (\xi < s)$$

The solution of Eq. (17) is

$$\bar{g}(s,\eta;\xi) = 0$$

$$= 1 - \tilde{F}(\eta) - \sum_{n=1}^{\infty} \hat{A}_n \left(\frac{s}{\xi}\right)^{-\gamma n/2} M_n(\eta) \tag{18}$$

where the coefficients $\hat{A}_n$ are selected so that $\bar{z}(s,\eta;\xi)$ is continuous at $\xi = s$; thus,

$$1 - \tilde{F}(\eta) - \sum_{n=1}^{\infty} \hat{A}_n M_n(\eta) = 0$$

and we have $\hat{A}_n = (C_n\gamma_n)^{-1}$, which is identical to the $\tilde{A}_n$ in Eq. (15).

By means of the Duhamel integral, the stagnation enthalpy $g(s,\eta)$ downstream from s_i can be written as

$$g(s,\eta) = g_w(s_i) + [1 - g_w(s_i)]\tilde{F}(\eta) + \int_{si}^{s} \bar{g}(s,\eta;\xi)\frac{dg_w(\xi)}{d\xi}d\xi$$

$$= g_w(s) + [1 - g_w(s)]\tilde{F}(\eta) - g_w(s)\left[\sum_{n=1}^{\infty}(C_n\gamma_n)^{-1}M_n(\eta)\right] +$$

$$\sum_{n=1}^{\infty}(2C_n)^{-1}M_n(\eta)s^{-\gamma n/2}\int_{si}^{s}\xi^{(\gamma n/2)-1}g_w(\xi)d\xi \tag{19}$$

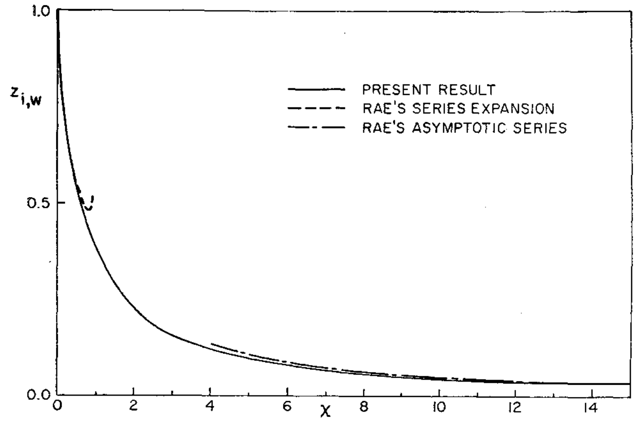

Fig. 3 Axial distribution of surface concentration, $\beta = 0.4$.

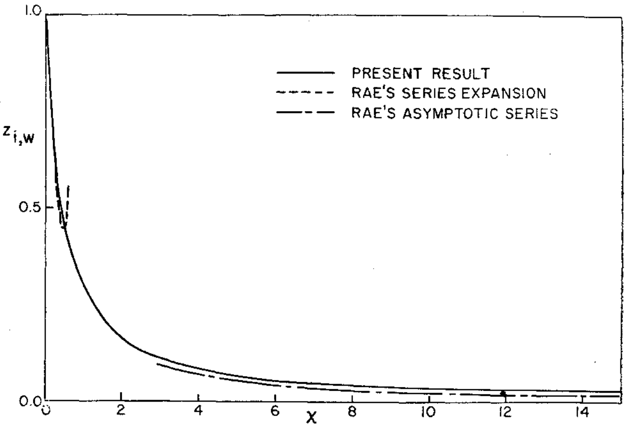

Fig. 4 Axial distribution of surface concentration, $\beta = -0.19$.

Since Eq. (4) is linear, the effects of specified initial profile, of arbitrarily distributed wall enthalpy, and even of the perturbation due to the deviation of C, σ, and of f can be treated by superposition. For more complicated wall conditions, we may construct an integral equation in terms of $g_w(s)$ as the dependent variable by substituting Eq. (19) into that wall condition. This actually is the method used for problems involving catalytic walls described below.

Wedge Flow with Catalytic Surface

We now employ the previously developed eigenvalues and eigenfunctions to the laminar boundary layer in a wedge flow with heterogeneous chemical reaction. If we make the further assumptions that the diffusion properties of the flow can be described by a single diffusion coefficient and that the Lewis number based thereon is unity, the continuity equation of species i can be written in terms of its mass fraction ratio, $Z_i = Y_i/Y_{i,e}$, and in terms of η and s as independent variables, i.e.,

$$Z_{i\eta\eta} + f_0 Z_{i\eta} - 2sf'_0 Z_{is} = 0 \tag{20}$$

where f'_0 is the solution of the Falkner-Skan equation.

The boundary condition at $\eta \to \infty$ can be written as

$$Z_i(s,\infty) = 1 \tag{21}$$

whereas the conditions at the catalytic surface require special discussion. In most of the aerospace literature only binary mixtures have been treated with the understanding that the analyses can be applied to the more complex, multicomponent flows, provided the reactants and products appear in highly diluted concentrations. The boundary conditions at the wall are then written in a simpler form[2]

$$Z_{i\eta}(s,0) = \zeta(s) Z_i^p(s,0) \tag{22}$$

where p is the reaction order and $\zeta(s) \equiv (\rho_w/\rho_e)^p k_w \rho_e^{p-1} (Sc/C)_w [(2s)^{1/2}/(u_e\mu_e)]$ is the so-called catalyticity. Libby and Liu[3] have generalized the problem by considering a boundary layer with Blasius solution as the velocity field undergoing surface catalysis with two reactants and two products according to a one-step, unidirectional reaction following a Langmuir-Hinshelwood mechanism. The boundary condition at the catalytic surface becomes much more complicated as shown in Eq. (7) of Ref. 3. They employ the corresponding eigenvalues and eigenfunctions to construct the solution for the mass fraction of species i which is identical to Eq. (19) of the present paper with the substitution of g by Z_i, where $Z_{i,w}(s)$ is considered to be unknown. Substitution of that mass fraction into the boundary condition at the catalytic surface leads to an integral equation that has $Z_{i,w}(s)$ as the dependent variable and can be solved numerically by iteration.

The analysis in Refs. 2 and 3 can be easily extended to cases in which the velocity field is described by the Falkner-Skan equation. As an example of this application of our eigenvalues and eigenfunctions to the species equation, we consider here a binary-mixture flow with reaction order $p = 1$. Let the catalyticity $\zeta(s) = \zeta_0 s^q$, where q can be associated with any power law behavior; thus the boundary condition at the wall is

$$Z_{i\eta}(s,0) = \zeta_0 s^q Z_i(s,0) \tag{23}$$

In the case of an incompressible laminar boundary layer of a very dilute reactant carried in an inert gas along the surface of a wedge, the power q is equal to $(1 - \beta)/2$ as indicated by Rae.[7] The initial condition at $s = 0$ can be determined by the ordinary differential equation obtained by substituting $s = 0$ into Eqs. (20, 21, and 23), such that

$$Z_i(0,\eta) = \left[1 - \frac{\bar{F}'(0)}{\zeta(0) + \bar{F}'(0)}\right] F(\eta) + \frac{\bar{F}'(0)}{\zeta(0) + \bar{F}'(0)} \tag{24}$$

Note that if $\zeta(0) = 0$, $Z_i(0,\eta) \equiv 1$, which is seen to be the case as long as $\beta \leq 1$.

For convenience in numerical calculation, we may replace the streamwise coordinate s by the new variable $\chi = \zeta_0 s^q$. Equation (20) then becomes

$$Z_{i\eta\eta} + f_0 Z_{i\eta} - 2q\chi f'_0 Z_{i\chi} = 0 \tag{25}$$

subject to the initial and boundary conditions

$$\begin{gathered} Z_i(0,\eta) = 1, \text{ provided } \zeta(0) = 0 \\ Z_{i\eta}(\chi,0) = \chi Z_i(\chi,0),\ Z_i(\chi,\infty) = 1 \end{gathered} \tag{26}$$

Following the same procedures as in Refs. 2 and 3, we construct a step-function solution corresponding to $\bar{g}(s,\eta;\xi)$ in Eq. (17), construct a solution of $Z_i(\chi,\eta)$ corresponding to $g(s,\eta)$ in Eq. (20), and substitute this form of $Z_i(\chi,\eta)$ into the boundary condition at the catalytic surface of Eq. (26); we get an integral equation of $Z_{i,w}(\chi)$, namely,

$$Z_{i,w}(\chi) = \left[\chi - \bar{F}'(0) + \sum_{n=1}^{\infty} (C_n\gamma_n)^{-1}\right] \times \left\{ -\bar{F}'(0) Z_{i,w}(0) + \frac{1}{2q} \sum_{n=1}^{\infty} C_n^{-1} \chi^{-(1/2q)\gamma_n} \times \int_0^{\chi} \xi^{(1/2q)\gamma_n - 1} Z_{i,w}(\xi) d\xi \right\} \tag{27}$$

In general this may be solved numerically by iteration. However, for a first-order reaction, we do not need iteration. If we denote by $Z_{i,w,k}$ the value of $Z_{i,w}(\chi_k)$, and $\chi_k = \chi_{k-1} + \Delta\chi$, where i indicates the order of points on the χ coordinate, and if the integral term is approximated by Simpson's rule, Eq. (27) can be presented as

$$Z_{i,w,k} = \left[\chi_k - \tilde{F}'(0) + \sum_{n=1}^{10} (C_n\gamma_n)^{-1} - \frac{\Delta\chi}{6q}\left(\sum_{n=1}^{10} C_n^{-1}\right)\chi_k^{-1}\right]^{-1} \cdot \left\{-\tilde{F}'(0)Z_{i,w}(0) + \frac{1}{2q} \times \sum_{n=1}^{10} \gamma_n\chi_k^{-\gamma_n/2q}\left[\int_0^{\chi_{k-2}} \xi^{(1/2q)\gamma_n - 1} Z_{i,w} d\xi + \frac{\Delta\chi}{3}\left(\chi_{k-2}^{(1/2q)\gamma_n - 1} Z_{i,w,k-2} + 4\chi_{k-1}^{(1/2q)\gamma_n - 1} Z_{i,w,k-1}\right)\right]\right\} \tag{28}$$

Equation (28) appears to have a singularity at the origin point on the χ coordinate, say $k = 1$ and $\chi_1 = 0$. However, for $\chi \approx 0$ we can express $Z_{i,w}$ in a series form, i.e.,

$$Z_{i,w}(\zeta) = Z_{i,w}(0) + a_1\chi^{\alpha_1} + a_2\chi^{\alpha_2} + \ldots + a_n\chi^{\alpha_n} + \ldots \tag{29}$$

Substitution of Eq. (29) into Eq. (27) yields

$$\alpha_1 = 1, \alpha_2 = 2, \ldots, \alpha_n = n, \ldots,$$

and

$$a_1 = Z_{i,w}(0) \bigg/ \left[\tilde{F}'(0) - q \sum_{n=1}^{\infty} (C_n\gamma_n)^{-1}\left(q + \frac{\gamma_n}{2}\right)^{-1}\right]$$

$$a_2 = a_1 \bigg/ \left[\tilde{F}'(0) - 2q \sum_{n=1}^{\infty} (C_n\gamma_n)^{-1}\left(2q + \frac{\gamma_n}{2}\right)^{-1}\right]$$

$$\ldots \tag{30}$$

$$a_k = a_{k-1} \bigg/ \left[\tilde{F}'(0) - kq \sum_{n=1}^{\infty} (C_n\gamma_n)^{-1}\left(kq + \frac{\gamma_n}{2}\right)^{-1}\right]$$

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With the numerical results of eigenvalues and their corresponding normalizing constants listed in Table 1, the a_n coefficients in Eq. (29) can be easily evaluated and we can calculate the first 2 points $Z_{i,w,1}$ and $Z_{i,w,2}$ by Eq. (29) whereas the further points can be calculated from Eq. (28) by a step-by-step method. Figures 3 and 4 show $Z_{i,w}$ as a function of χ for $\beta = -0.19$ and $\beta = 0.4$. The numerical values of $Z_{i,w}$ for $0 < \chi < 0.1$ are evaluated from the series solution of Eq. (29) whereas those for $\chi > 0.1$ are calculated from Eq. (28). We also have plotted approximate solutions given by the power series as well as the asymptotic series of Rae (Ref. 7) for comparison. We see that our present results agree in the two limits of χ small and large with Rae's results and provide a continuous distribution of $Z_{i,w}$ between the limits.

Concluding Remarks

We have developed eigenfunction solutions for the nonsimilar equations of energy and species conservation for boundary layer flows with velocities described by Falkner-Skan solutions and have applied them to two problems, one concerned with heat transfer, the other with surface reactions. From previous applications of the related analyses for the special case of $\beta = 0$, i.e., if the pressure gradient is zero, it is clear that a variety of interesting problems can be treated in terms of the eigenfunctions presented here.

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